Sources and sinks separating domains of left- and right-traveling waves: experiment versus amplitude equations

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In many pattern forming systems that exhibit traveling waves, sources and sinks occur which separate patches of oppositely traveling waves. We show that simple qualitative features of their dynamics can be compared to predictions from coupled amplitude equations. In heated wire convection experiments, we find a discrepancy between the observed multiplicity of sources and theoretical predictions. The expression for the observed motion of sinks is incompatible with any amplitude equation description.

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Since its inception [1], the amplitude equation approach has grown out to become an important organizing principle of the theory of non-equilibrium pattern formation — it has not only enabled us to uncover a number of general features of near-threshold pattern dynamics, but has also allowed us to understand the influence of boundary conditions, defects, etc. Many qualitative and quantitative predictions have been successfully confronted with experiments [2]. The most detailed comparison with experiments has been made for the type of systems for which the theory was originally developed, hydrodynamic systems that bifurcate to a stationary periodic pattern. For traveling wave systems, the range of validity of the appropriate amplitude equation is, however, much more an open question, both because the theoretical derivation has been performed for only a few systems [3], and because direct tests are difficult. Moreover, in practice complications often arise due to the presence of additional important slow variables [4].

It is the aim of this paper to point out that sources and sinks that separate patches of traveling wave states, can provide a clear way of testing the consistency of the experimental observations with *generic qualitative* predictions from amplitude equations. Sources and sinks are distinguished by whether the group velocity points out- or inwards — see Fig. 1. They occur in a wide variety of systems where oppositely traveling waves suppress each other — in directional solidification [5], the printer instability [6], eutectic growth [7], as well as in convection [8,9] — but their properties have remained largely unexplored. We illustrate the idea we put forward with experiments on traveling waves occurring in a liquid heated by a wire just below the surface [8–10]. In the parameter range we have been able to explore (dimensionless control parameter $0.25 \lesssim \varepsilon \lesssim 0.5$), the experimental properties of sources and sinks we observe are *inconsistent* with the behavior predicted by the standard coupled amplitude equations for the near-threshold behavior in a one-dimensional system with left- and right-traveling waves [2] ,

$$(\partial_t + s_0\partial_x)A_R = \varepsilon(1+ic_0)A_R + (1+ic_1)\partial_x^2 A_R - (1-ic_3)|A_R|^2A_R - g_2(1-ic_2)|A_L|^2A_R \ , \qquad (1a)$$

$$(\partial_t - s_0\partial_x)A_L = \varepsilon(1+ic_0)A_L + (1+ic_1)\partial_x^2 A_L - (1-ic_3)|A_L|^2A_L - g_2(1-ic_2)|A_R|^2A_L \ . \qquad (1b)$$

In these equations, we have used suitable units of space and time, and A_R is the amplitude of the right traveling mode $e^{-i(\omega_c t - k_c x)}$ and A_L the one of the left traveling mode $e^{-i(\omega_c t + k_c x)}$. Furthermore ε is the control parameter which measures the distance from the threshold of the instability at $\varepsilon = 0$, and the parameters $c_0 - c_3$ are related to the linear (c_0, c_1) and nonlinear (c_2, c_3) dispersion of the waves. It is well-known that, strictly speaking, the above equations only arise as the lowest order amplitude equations if the linear group velocity s_0 is of order $\varepsilon^{1/2}$; in practice, the equations are often also applied to cases in which s_0 is finite at threshold, for lack of a good alternative. This amounts to the idea that since (1) includes all the necessary terms and respects all the proper symmetries, it is not unreasonable to hope that these equations still provide a good qualitative description outside their proper range of validity [11].

When the coupling parameter g_2 in (1) is larger than 1, the left- and right-traveling waves suppress each-other [2], and the system evolves to a state consisting of patches where either A_L or A_R is zero. Within such a patch, a *single* amplitude equation suffices, and the group velocity term can be removed by a Galilean transformation. While many experimental and theoretical studies have focused on this situation, we wish to study the sources and sinks that connect these patches of left- and right-traveling waves (see Fig. 1). These coherent structures involve both amplitudes A_R and A_L and therefore the group velocity terms can not be removed. A study of their properties may shed some light on the applicability of (1) to real patterns.

Our experimental set-up, shown in Fig. 2, is a simple system based on a electrically heated wire immersed below the surface of a fluid [8–10]. Beyond a critical heating

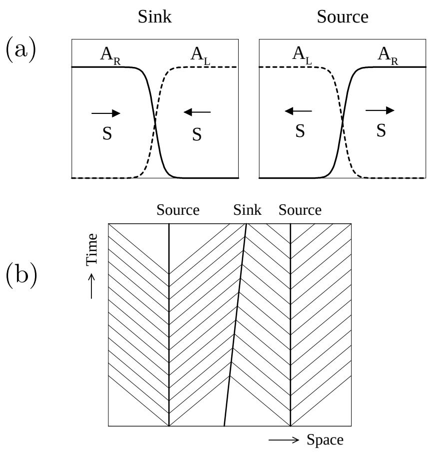

FIG. 1. Definition of sources and sinks. *(a)* Illustration of sources and sinks as coherent structures in terms of the behavior of the amplitudes A_L and A_R of the left- and right traveling waves near these structures. A source is defined as a coherent structure at which waves with total group velocity s pointing outward are generated, a sink one at which waves with group velocity s pointing inwards annihilate each other. *(b)* Illustration of the kinematics of sinks and sources in terms of the properties of the adjacent waves, for the case that the group velocity s has the same sign as the phase velocity (as in the experiment, where $s \approx v_{ph}/3$). In *(b)*, the definition of source and sinks given under *(a)* agrees with the intuitive notion that the waves travel away from a source and into a sink. In this figure, the thin lines indicate lines of constant phase of the traveling waves. In accord with our experimental obervations, illustrated in Fig. 3b, we have drawn a case with two stationary and symmetric sources, each generating waves with different wavenumbers and frequencies, and one sink moving according to the phase matching rule. According to this rule, every constant phase line coming in at the source from the left matches up with a constant phase line coming in from the right: Phases at the sink match. A simple geometric construction gives the velocity of a sink at which phases match in terms of the frequencies and wavenumbers of the incoming waves — see Eq. (2).

power Q_c, traveling periodic modulations appear at the surface via a forward Hopf bifurcation [8,9]. The apparatus is similar to the one used in [8,9] but the design of the Plexi-glass cell is somewhat different and larger (55x15x6 cm), so that the sides are further away from the wire. Both top to bottom and lateral views are possible in our set-up; in particular, the lateral view proved especially useful for recording the time series signals. A tungsten wire with a diameter of 0.1 mm is heated by means of an electrical current and immersed in oil in the middle of the cell and parallel to the longest side. The viscosity of the GE SF 96 silicone oil is 0.5 stoke. Both the voltage across the wire and the heating current were continuously monitored to check that their values did not change during the measurements. Four springs (two on each end of the cell) provide the necessary tension to keep the wire parallel to the surface of the fluid throughout the cell. A pair of micrometers attached to the ends of the cell enables us to carefully adjust the wire-surface distance. A shadowgraph technique is used to record the signal generated by the waves. The cell is illuminated with unpolarized white light. Two photodetectors can be placed at adjustable positions along the wire, and the light signal captured by the detectors is sent to a digital oscilloscope. This temporal signal has a very local character thanks to the diverging geometry from the light source towards the acquisition plane. It therefore allows us to measure the local frequency very accurately, even though the relation between the signal and the surface modulations is quite nonlinear due to the strongly inhomogeneous temperature distribution in the direction perpendicular to the wire. Since a single measurement may take several hours due to the typical long times that fluids need to reach a steady state condition, the temperature of the surrounding must be controlled. For this reason, air conditioning was steadily supplied and the temperature was continuously monitored.

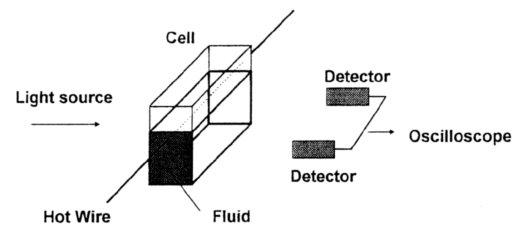

FIG. 2. Schematic drawing of the experimental setup. The frequencies of the traveling waves can be probed at two positions with photo-detectors. The distance between the surface of the liquid and the wire was varied from 1 to 3 mm.

The picture of the typical sequence of events can be drawn as follows [8–10]. Having chosen an adequate depth for the wire, domains of left- and right-traveling waves emerge after the power Q is turned on. These patches are separated by sinks and sources. Sources in our experiment send out waves to both sides, while sinks have oppositely traveling waves coming in from both sides. Once transients have died out, the sources stay at some fixed position while the sinks generically move, either towards a source (in which case they usually annihilate each other) or a boundary (thus also disappearing from the scene). A typical example is shown in Fig. 3b. The time that a simple state, say one with two or three sources and sinks, remains in the cell is quite arbitrary; in the end a source usually is the longest living object [9,10]. We have mainly explored the range $0.25 \lesssim \varepsilon \lesssim 0.5$ for the control parameter $\varepsilon \equiv Q/Q_c - 1$. A space-time plot of a source solution in this regime is shown in Fig. 2 of [9], while sideways snapshots of regions of the cell with a source and a sink are shown in Figs. 3c and 3d.

Our main *experimental observations* concerning the dynamics of sources and sinks are the following:

(i) The relative motion of sinks and sources is independent of their separation, and so there does not appear to be a long-range interaction between them.
(ii) Sources always have zero velocity, $v_{so} = 0$, and are symmetric: the wavenumber and frequency of the outcoming left-traveling mode are always the same as those of the outcoming right-traveling mode. The data that illustrate the stationarity of the source are shown in Fig. 3b, while the fact that the waves emerging from a source are symmetric is illustrated by the photo-detector data shown in Fig. 3a. In this figure, the frequencies of the signals form the detectors D_1 and D_2 at both sides of the left source are exactly the same, and so are those of the signals taken at opposite sides of the source on the right by detectors D_3 and D_4.
(iii) While sources are stationary and symmetric, they are not unique: each source sends out waves with a well-defined frequency and wave number, but different sources send out different waves — compare, e.g., the two sources of Fig. 3a: the frequency of the signals of D_1 and D_2 is different from that of D_3 and D_4. We take this as evidence that in these experiments at least a one-parameter family of sources exists.
(iv) As Fig. 3b illustrates, sinks typically move. Moreover, most of our sinks are found to move in such a way that the incoming phases *match* at the sink [12]: in the frame traveling with the sink, the frequencies of the waves coming in from both sides are equal and no phase difference builds up across these sinks. This was already illustrated in Fig. 1b. If we write the two appropriate incoming modes as $e^{-i(\omega_R t - ik_R x)}$ and $e^{-i(\omega_L t + ik_L x)}$, then the velocity v_{si}^{match} of such a sink is immediately found to be

$$v_{si}^{match} = \frac{\omega_R - \omega_L}{k_R + k_L} \; . \tag{2}$$

This relation implies that when a sink is sandwiched between two sources, it moves away from the source with the largest frequency and its velocity is completely determined by the properties of the adjacent sources.

We now confront these results with *theoretical predictions*. Since the pattern occurs via a forward Hopf bifurcation [8,9], the generic amplitude equations are given by Eqs. (1). We take $g_2 > 1$ since traveling wave states occur, and since the group velocity is is about a third of the phase velocity in this experiment, we allow the linear group velocity s_0 to be of order 1. Note that for our analysis, we do not need knowledge of the values of the other parameters occuring in (1) [13].

Property *(i)*, the absence of long-range interactions between sources and sinks, is consistent with the fact that amplitudes at both sides of source and sink solutions approach their asymptotic value exponentially fast, as in the single-mode equation, Eq. (1a) with $A_L = 0$ [15,16].

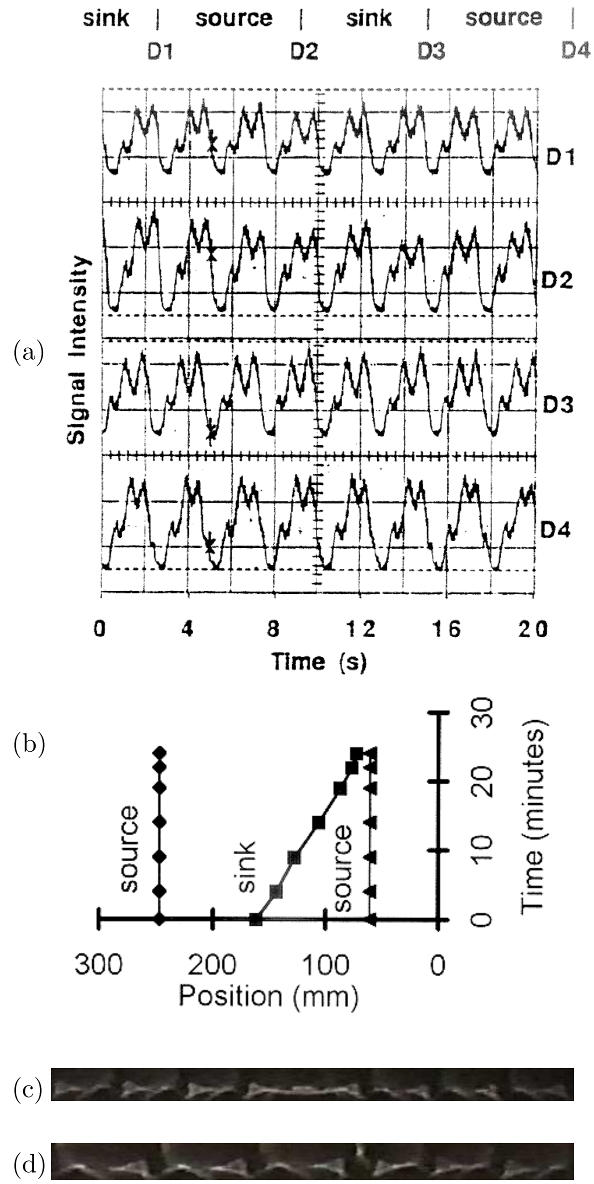

FIG. 3. Experimental results. *(a)* Example of the signals from the photodetectors placed at various positions in between source and sink solutions, as indicated. Typical signal amplitudes are about a factor 8 above the noise level. The absolute value of each signal is arbitrary, especially the frequency is relevant. Note that the frequencies of the two waves sent out by each separate source are exactly the same, but that the two sources send out different waves. As a result, the sink is sandwiched in between different incoming waves. *(b)* Example of traces of the position of sinks and sources in the experiment. Compare Fig. 1b, where a similar situation is drawn schematically. *(c,d)* Snapshots of regions of the experimental cell with a source (*c*) and a sink (*d*), taken from a sideways video image of the cell. Note the asymmetry of the pattern around the source, which is roughly in the middle of *(c)*, and the asymmetry of the pattern around the sink, which is slightly right of center in *(d)*.

To compare with observations *(ii)* and *(iii)*, we have

analyzed the generic existence and multiplicity of *source* solutions of (1) with an extension of previous counting arguments [15] for solutions of the form $A_R = e^{-i\omega_0 t}\hat{A}_R(x - v_{so}t)$ and $A_L = e^{-i\omega_0 t}\hat{A}_L(x + v_{so}t)$. Our analysis [17] shows that independent of the specific values of the parameters, source solutions of (1) *generically* come in discrete sets. In particular, one typically expects there to be only a *unique* symmetric source solution with $v_{so} = 0$, and numerical simulations of (1) confirm this. This is in clear contradiction with the experiments, where we find a continuous family of them!

We now turn to *sinks*, which according to *(iv)* move in the experiments with a velocity (2) [14]. Can the phase matching property of the sinks underlying this equation be reproduced at all in an amplitude approach based on (1)? The answer is *no*. To see this, note that v_{si}^{match} is according to (2) given in terms of the *total* frequencies ω_R, ω_L and wave numbers k_R, k_L of the incoming modes. In an amplitude expansion, these are written as an expansion about their critical values, e.g., $\omega_R = \omega_c + \omega_{A_R}$ where ω_{A_R} is the frequency of the amplitude A_R of the right-traveling mode, etc. In terms of these, the experimentally observed velocity of phase-matching sinks becomes

$$v_{si}^{match} = \frac{\omega_{A_R} - \omega_{A_L}}{2k_c + k_{A_R} + k_{A_L}} \ , \qquad (3)$$

which underscores once more the fact that this velocity depends on both the fast and the slow spatial scales. However, as an amplitude description is based on an adiabatic elimination of the fast scales, the amplitude equations (1) do not involve the parameters ω_c and k_c associated with the fast scales. So, although families of moving sink solutions do exist for (1), there is no mechanism in these equations to single out the velocity (3) as the selected velocity of sinks.

In summary, our results demonstrate that generic properties of sources and sinks in traveling wave systems, like their multiplicity and dynamics, allow a simple yet powerful comparison between experiments and amplitude equation descriptions. For the heated wire convection experiment in the range $0.25 \lesssim \varepsilon \lesssim 0.5$ our experiments are inconsistent with an amplitude description. Although a final conclusion must await further study of the $\varepsilon \to 0$ limit, our results point to two important issues. First of all, they question the soundness of using Eqs. (1) for systems with finite group velocity s_0. Secondly, they provide a clear example of the possible importance of non-adiabatic effects (coupling of the slow and fast scales [18]) in sinks, even though the two are widely separated, as the relative frequency modulation $\Delta\omega/\omega_c$ (which is comparable to the ratio of the typical sink velocity and the phase velocity) can be as small as 1/50 in our experiments.

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[13] Although we don't need this in the analysis, the stability of the traveling waves necessitates to take $c_1 c_3 < 1$ [2].

[14] When the phases match across a sink, one might be tempted to say that no phase-slips occur there. We prefer to avoid to use the word phase-slip in this context, however, as it usually refers to events at which the amplitude goes through zero in a single amplitude equation [15]. In the case of two coupled modes, each separate amplitude does not need to exhibit phase slip events even if the phases across a sink do not match. The relation between phase-slips of the amplitudes and of the patterns is quite different for single and double amplitude equations.

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